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Ramsey numbers involving a triangle: theory and algorithms

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Ramsey Numbers Involving a Triangle: Theory & Algorithms

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To my dear Grandma

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Title of Thesis: Ramsey Numbers Involving a Triangle: Theory and Algorithms.

ABSTRACT

Ramsey theory studies the existence of highly regular patterns in large sets of objects. Given two graphs G and H, the Ramsey number $R(G,H)$ is defined to be the smallest integer n such that any graph F with n or more vertices must contain G, or $\bar{F}$ must contain H. Albeit beautiful, the problem of determining Ramsey numbers is considered to be very difficult.

We focus our attention on efficient algorithms for determining Ramsey numbers involving a triangle: $R(K_3, G)$. With the help of theoretical tools, the search space is reduced by using different pruning techniques and linear programming. Efficient operations are also carried out to mathematically "glue" together small graphs to construct larger *critical* graphs.

Using the algorithms developed in this thesis, we compute all the Ramsey numbers $R(K_3, G)$, where G is any connected graph of order seven. Most of the corresponding critical graphs are also constructed. We believe that the algorithms developed here will have wider applications to other Ramsey-type problems.

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1. Introduction

Ramsey theory originated in the work of Frank Plumpton Ramsey, a mathematician at the University of Cambridge in England, who was keenly interested in mathematical logic, philosophy, and economics. Although only 26 years old when he died in 1930, Ramsey left behind a rich legacy of mathematical results that continue to intrigue mathematicians. One of his arguments concerning predicate logic led itself to a fascinating problem, which is now easily interpreted as "party problem" (however seldom used by any party planner) — how many people do you need to invite so that you can guarantee that there are either m mutual acquaintances or n mutual strangers? Mathematicians often refer to its solution as the *Ramsey number* $R(m, n)$.

Due to its beauty, as well as complexity and difficulty, Ramsey theory and Ramsey numbers have been attracting great attention in the vast amount of graph theoretical problems. Since its start with Ramsey's original paper [Ram], there has been a rich literature on Ramsey type problems. In particular, Graham, Rothschild and Spencer in their book [GRS] presented an exciting development of Ramsey theory. Although mathematicians have obtained many results with regard to asymptotic bounds for various types of Ramsey numbers, progress on the evaluation of Ramsey numbers seems far from satisfactory. Moreover, these asymptotic results yield typically loose estimates.

However, with the assistance of computers, considerable progress has been made in the past decade to evaluate small Ramsey numbers by designing new algorithms on the proper theoretical basis. The results described in this paper are also of this nature. We believe that this approach will be effective for other small Ramsey numbers to either improve bounds or obtain certain exact values.

1.1. Overview of thesis

The rest of Section 1 introduces some basic terminology in graph theory that will be used subsequently. The meaning of a Ramsey number is also described.

Starting from Section 2, the reader will be led to a beautiful world of Ramsey theory. In Section 2 we discuss the importance of Ramsey theory, and present a survey of the progress in Ramsey theoretic problems. Some basic properties regarding Ramsey numbers are mentioned. Our motivation for this study will also be given there.

Our main results, all Ramsey numbers for a triangle versus any connected graph on 7 vertices will be presented in Section 3. In Section 4 we will design, via both theoretical deduction and algorithms, the process by which the results were obtained. In Section 5 readers will see some interesting computational techniques and tools used in this work. We will also present other related test cases to verify our main results.

Discussions in Section 6 will point out some easy but important related consequences. Further improvements and extensions will also be given.

To make the thesis succinct, most of the theorems which can be found in textbooks or other papers are only cited with references and with their proofs omitted. We remark here that in the cases where any non-trivial lemma or theorem is not credited, the result is original.

1.2. Basic concepts

Graph theoretical concepts are important here. Although you do not have to be an expert in graph theory before attempting to solve Ramsey type problems, knowledge in general graph theory is extremely helpful (A good introduction is [Hal]). Also, logical reasoning is heavily used in this thesis, and readers are assumed to have basic knowledge of mathematical logic and set theory. We also recommend that reader be familiar with general concepts in data structures and algorithms.

Graphs

A graph G is a system consisting of a vertex set V and an edge set E, denoted by (V, E). V, E are also sometimes denoted by $V(G)$ and $E(G)$, respectively. Graphs having loops or multiple edges between two vertices are not studied here. The size of V is called the *order* of graph G, denoted as $n(G)$. We also use $e(G)$ to represent the number of edges in G. Sometimes we also label the vertex set V with natural numbers, yielding a *labeled graph.* In this thesis we are only interested in combinatorial structure of graph, not its possible drawings. Hence if the vertex set of a graph is labeled in two different ways, yielding G_1 and G_2, we will regard them to be of the same structure, and say that G_1 is *isomorphic* to G_2.

Given a graph $G = (V, E)$, its *complement*, $\overline{G}$, has the same vertex set, but the edge set consists of exactly the non-edges in G. Also, a graph $G = (V, E)$ can have *subgraph* $G' = (V', E')$, if $V' \subseteq V$ and $E' \subseteq E$. In this case, we usually say that graph G *contains* graph G', and write $G' \subseteq G$. Furthermore, if for every $G'' = (V', E''), G'' \subseteq G$, G'' is a subgraph of G', we say that G' is the subgraph of G *induced* by V', denoted by $G[V']$.

The *degree* of a vertex v in G, $deg(v)$, refers to the number of other vertices in G connected to v by an edge. The *minimum degree* of G, $\delta(G)$, is the minimum of $deg(v)$, where v is any vertex in $V(G)$. Let $X \subseteq V$ be a vertex set, we also define $N_X(v)$ to be the subset of X in which every vertex is connected to v.

The *complete graph* on n vertices, denoted K_n, is the one in which every pair of vertices is joined by an edge. A complete subgraph of a graph is referred to as a *clique*. The complement of a complete graph is caled an *independent set*. A graph is *triangle-free* if it contains no triangles; i.e., K_3 is not a subgraph.

The *chromatic number* of a graph is defined to be the smallest number of colors that must be used to color the vertex set such that no two vertices can have the same color if there is an edge between them.

The list in Table 1.1 defines some graphs which will be referred to later in this thesis. Other graph theoretical terms, which are more closely related to Ramsey theory, will be introduced in later sections.

Notation	*Term*	*Definition*
$K_n - e$		K_n missing an edge
$K_n - iK_2$		K_n missing i disjoint edges
P_n	*path*	a path on n vertices
C_n	*cycle*	a cycle of length n
W_n	*wheel*	graph formed by some vertex x connected to all vertices of some cycle C_{n-1}
T_n	*tree*	a connected graph on n vertices with no cycles
$K_{m,n}$	*complete bipartite*	K_{m+n} missing disjoint K_m and K_n

Table 1.1. Notation

Ramsey numbers

Given two graphs G, H, the (generalized) Ramsey number $R(G, H)$ is defined to be the minimum number of vertices in a graph F such that no matter what F is, either F has a subgraph isomorphic to G or $\overline{F}$ has a subgraph isomorphic to H. In other words, as long as a graph reaches a certain number of vertices, either itself or its complement cannot avoid certain substructures. The question is to find out what that critical "boundary" is.

If a graph F on n vertices and e edges does not have subgraph G, and its complement $\overline{F}$ does not have subgraph H, F is said to be (G, H, n, e)-good. (n and e are sometimes omitted.) $e(G, H, n)$ means the minimum number of edges of all (G, H, n)-good graphs. If $n = R(G, H) - 1$, any (G, H, n)-good graph is called a *critical* (G, H)-good graph.

We use notation $R(k, H)$ and $R(G, k)$ for $R(K_k, H)$ and $R(G, K_k)$, respectively. $R(G)$ is defined to be the diagonal case $R(G, G)$. The classical Ramsey numbers refer to the case of complete graphs $R(K_m, K_n)$, denoted also by $R(m, n)$, where m and n are positive integers.

2. Previous work

There is a vast literature on Ramsey type problems, starting in 1930 with the much celebrated paper of Ramsey [Ram]. Here we restrict our attention to a graph theoretical point of view, concentrating basically on evaluating exact values of small graph Ramsey numbers.

2.1. Why study Ramsey theory?

The study of Ramsey numbers and Ramsey graphs originates from pure mathematics. It is strongly related to design theory, graph theory, and arithmetic progression analysis. With the development of the Ramsey theory, people are discovering more and more fascinating applications, such as in the area of information retrieval [Yao], communications [Lov] and decision making [KT].

2.2. Historical overview

To evaluate the exact value of a Ramsey number is difficult. The general approach is to achieve better estimate of upper and lower bounds, in the hope that the two bounds will meet after many sleepless nights of mathematicians and computer scientists. The surveys by S. A. Burr [Bu1] and T. D. Parsons [Par] contain extensive chapters on general exact results in graph Ramsey theory. F. Harary presented the state of the theory in 1981 in [Ha3], where he also gathered many references, including seven to other survey papers. A decade ago, Chung and Grinstead in their survey paper [ChGri] included a broad discussion of different methods used in Ramsey computations in the classical case. S. A. Burr, one of the most experienced researchers in Ramsey graph theory, formulated in [Bu3] seven conjectures on Ramsey numbers for sufficiently large and sparse graphs, and reviewed the evidence for them found in literature.

Newer extensive presentations can be found in [GRS] and [GrRö], though these focus on asymptotic theory, not on the numbers themselves. *Journal of Graph Theory* [JGT, 1983] dedicated an entire volume to Ramsey theory. Besides a number of research papers, it included historical notes and presented to us Frank P. Ramsey (1903-1930) as a person. Radziszowski recently made an extensive compilation [Ra2] of all non-trivial known values and bounds for classical Ramsey numbers, as well as other references to general Ramsey numbers, and 9 survey papers.

The most studied cases, and the hardest, are classical Ramsey numbers. Only 9 non-trivial classical Ramsey numbers have been obtained so far, listed in Table 2.1 together with known non-trivial bounds for some other cases. The lower left triangle in Table 2.1 is empty because of symmetry $R(n,m) = R(m,n)$, whereas the lower right corner is empty because no substantial work has been published.

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
3	*6*	*9*	*14*	*18*	*23*	*28*	*36*	*40* *43*	*46* *51*	*51* *60*	*59* *69*	*66* *78*	*73* *89*
4		*18*	*25*	*35* *41*	*49* *62*	*53* *85*	*69* *116*	*80* *151*	*93* *191*	*97* *238*	*112* *291*	*119* *349*	*121* *417*
5			*43* *49*	*58* *89*	*76* *145*	*95* *219*	*320*						
6				*102* *165*	*304*	*499*	*786*						
7					*205* *549*	*1048*	*1783*						
8						*282* *1896*	*3675*						
9							*565* *6721*						
10								*798*					

Table 2.1 Nontrivial values and bounds
for Ramsey numbers $R(m,n)$

Chvátal and Harary [CH1,CH2] formulated several simple but very useful observations on how to discover values of some numbers. In 1977 Clancy [Clan] produced a widely cited table of $R(G,H)$ for all graphs G on at most 4 vertices and H on 5 vertices, except five entries. Four of the five missing entries in Clancy's table were found by 1989, and the last case was finally solved by McKay and Radziszowski in 1993 [MR], thus completing that table.

The subject itself has grown amazingly, in part by studying asymptotic behavior and generalizing the classical case to various graph structures such as $K_n - e$ ([BH] [Clan] [CEHMS] [CH1] [CH2] [EHM1] [FRS] [GH] [Ra1]), paths ([GeGy]), cycles ([BES] [FS] [MiSa] [Ros]), wheels ([RJ] [FM] [He]), trees ([EG] [GRS] [FSS]), bipartite graphs ([BES] [EHM2] [Ha2]) and others. Below we concentrate on results for triangle-free graphs, i.e., on the case of $R(K_3, G)$.

The study of minimum number of edges in a triangle-free graph on n vertices with no independent set of size k, $e(3,k,n)$, and the construction of some related minimum graphs enabled, in particular, the evaluation of the exact values of $R(3,6)$ [Ka, 1966], $R(3,7)$ [GY, 1968] and $R(3,9)$ [GR, 1982].

Radziszowski and Kreher presented a method using computers to construct all $(3,6)$-good graphs in [RK1], and later proved in [RK2] that $e(3,k+1,n) \geq 6n - 13k$.

McKay and Zhang used various computer algorithms to disprove the existence of $(3,8,28)$-good graphs, and established that $R(3,8) = 28$ [MZ, 1992].

2.3 Ramsey theorems

The existence of Ramsey numbers was proved by Ramsey [Ram], whose paper was originally concerned with applications to formal logic.

Theorem 2.1(Ramsey's Theorem, 1930). *If p and q are integers with $p, q \geq 2$, then there is a positive integer N that every graph F on N vertices either contains K_p, or $\overline{F}$ contains K_q.*

Here we enumerate some basic properties of Ramsey numbers. Interested readers are encouraged to refer to [GRS] for more in depth discussion.

Theorem 2.2. $R(G,H) = R(H,G)$.

It is this theorem that permits us not to distinguish between $R(G,H)$ and $R(H,G)$. To facilitate the presentation of this work, readers will find in later sections that both notations $R(3,G)$ and $R(G,3)$ are used. It is important to observe that the *complement* of any (G,H)-good graph is (H,G)-good, and the complement of *critical* (G,H)-good graph is critical for $R(H,G)$.

Theorem 2.3. (Erdös and Szekeres[1935] and Greenwood and Gleason [1955]) *For all integers* $p, q \geq 3$,

$$R(p,q) \leq R(p,q-1) + R(p-1,q),$$

and *if the two terms* on *the right hand side are* even, *this inequality is strict.*

This theorem has been used to estimate the upper bounds of Ramsey numbers for large p and q. It also gives a proof of Ramsey's Theorem. The argument is by induction on $k = p + q$.

We now concentrate on evaluating Ramsey numbers for a complete graph versus any graph structure. The following very interesting result in graph Ramsey theory was obtained by Chvátal and Harary:

Theorem 2.4. [CH2] *If* G *is a connected graph on* n *points, then*

$$R(k,G) \geq (k-1)(n-1) + 1. \tag{1}$$

This theorem establishes a lower bound of Ramsey number for a complete graph versus a connected graph. Bearing this in mind, we say that graph G is k*-good* when equality holds in (1). Another important result related to k-goodness is

Theorem 2.5. [Bu2] *For any connected graph* G *with* n *points,*

$$R(k,G) \geq R(k-1,G) + n - 1.$$

Theorems 2.4 and 2.5 have an immediate consequence that if a connected graph G is k-good, G is also $(k-1)$-good.

Chvátal [Chv] obtained that a tree T on n points satisfies $R(k, T_n) = (k-1)(n-1)+1$. In other words, trees are k-good for all k. Burr and Erdös showed in [BE] that wheels with at least 6 vertices are 3-good. Our previous results in [RJ] not only proved that paths, cycles and wheels on n vertices all have triangle Ramsey number $2n-1$ ($n \geq 6$ for wheels), but also characterized and counted all the corresponding critical graphs.

We also mention the following extensions of (1), which can lead to a broader concept of goodness of a graph:

Theorem 2.6. [BEFRSGJ] *For any connected graph* G, $R(F,G)$ *satisfies*

$$R(F,G) \geq (\chi(F)-1)\,(n(G)-1) + s(F), \qquad \textit{if} \quad n(G) \geq s(F).$$

where $\chi(F)$ is the chromatic number of F, and $s(F)$, the chromatic surplus, is the smallest number of vertices in a color class under any $\chi(F)$-coloring of $V(F)$.

In 1980, Faudree, Rousseau and Schelp determined Ramsey numbers $R(K_3, G)$ for all connected graphs G of order 6, which are summarized in Theorem 2.7. Our results, as will be presented in Section 3, go beyond this.

Theorem 2.7. [FRS] *With the graphs G_0 through G_{12} as identified by their complements in Figure 2.1, the following table gives $R(3, G)$ for every connected graph G of order 6.*

Class	G	$R(3, G)$
(a)	$G \subseteq G_0$	11
(b)	$G_1, \cdots, G_6$	12
(c)	G_7, G_8	13
(d)	$G_9, \cdots, G_{12}$	14
(e)	$K_6 - e$	17
(f)	K_6	18

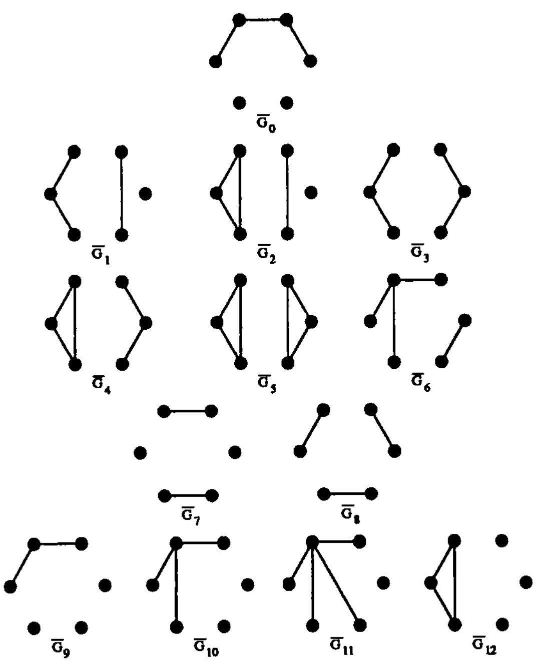


Figure 2.1

From these, the following results can be assembled:

Theorem 2.8. (easy) *For a connected graph G, $n(G) \le 6$, G is 3-good if any only if G is a subgraph of $K_2, K_3 - P_2, K_4 - P_2, K_5 - P_3$, or $K_6 - P_4$, with G on $2, 3, \cdots, 6$ vertices respectively.*

These results indicate that there are many families of graphs that are 3-good. A related result was obtained by Sidorenko [Sid], where he proved that $R(3, G) \le 2e(G) + 1$, and conjectured $R(3, G) \le n(G) + e(G)$.

One direction in the characterization of graph's k-goodness is by using the ***edge density*** of a graph G, defined as

$$\max_{F \subseteq G} \frac{e(F)}{n(F) - 1},$$

where the maximum is over all subgraphs of G. Burr and Erdös conjectured in [BE] that for any fixed k and x, all sufficiently large connected graphs with edge density of no more than x are k-good.

Another approach is to study the number of edges and chromatic numbers for those graphs. Let $f(k, n)$ be the largest e such that every connected graph on n vertices and e edges is k-good, and $g(k, n)$ be the largest e such that there exists a connected graph on n vertices and e edges to be k-good. We list the known values of f and g for $k = 3$ below in Table 2.2.

n	$f(3,n)$	$g(3,n)$
3	2	2
4	5	5
5	7	8
6	8	12
7	11	16
8	?	?

Table 2.2. Known values of $f(3, n)$ and $g(3, n)$.

It is easy to obtain $f(3, n)$ and $g(3, n)$ for n up to 5. $f(3, 6)$ and $g(3, 6)$ follow from Theorem 2.7. $f(3, 7)$ and $g(3, 7)$ are determined in this thesis in Section 6. We encourage interested readers to think about $f(3, 8)$ and $g(3, 8)$.

The authors of [BEFRS] also obtained some asymptotic bounds for f and g, which are summarized in Theorem 2.9 and Theorem 2.10.

Theorem 2.9. [BEFRS] *If $n \geq 4$, then*

$$f(3,n) \geq \frac{1}{15}(17n+1).$$

If $\epsilon > 0$ is fixed, then if n is sufficiently large, then

$$f(3,n) < (\frac{27}{4} + \epsilon) n \log^2 n.$$

Theorem 2.10. [BEFRS] *There are constants C and D such that*

$$Cn^{3/2} \log^{1/2} n < g(3,n) < Dn^{5/3} \log^{2/3} n.$$

Harborth and Mengersen also studied Ramsey numbers for sets of graphs with fixed number of edges. Let $R_{m,n}(s,t)$ be the smallest r such that every graph of order r must contain any graph with m vertices and s edges, or its complement must contain any graph on n vertices and t edges. They tabulated $R_{3,7}(s,t)$ in [GH] and studied $R_{n,n}(s,t)$ in [HM].

3. New results

We summarize in this section the results of our computation for Ramsey numbers $R(3, G)$, where G is any connected graph on seven vertices. In Theorem 3.1, classes of graphs will be described, with each member of a given class having the same Ramsey number. In particular, we identify each class by its ***top*** graph set T and ***bottom*** graph set B. All graphs contained in any one of the top graphs, and also containing any one of the bottom graphs at the same time, belong to the same class.

For example, Figure 3.1 lists all graphs on 4 vertices. If we consider the containment relation (as identified by a dotted arrow), all graphs enclosed in the dotted line area (and no other graphs) form a class, which can be defined by top graph set $T = \{G_1\}$ and bottom graph set $B = \{G_2, G_3\}$. Incidentally, they all have Ramsey number $R(3, G) = 7$. Note that $R(3, G)$ does not increase as we go down in the "net".

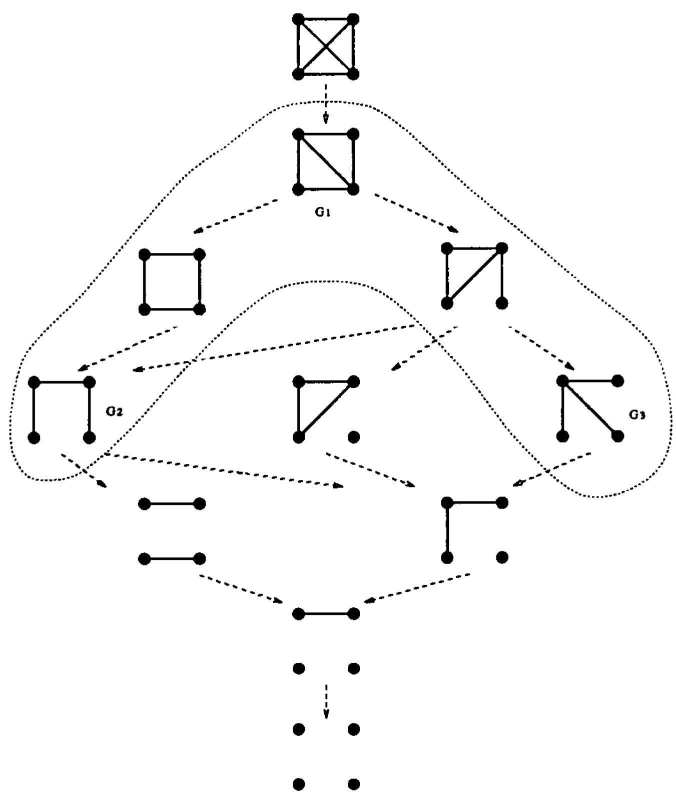
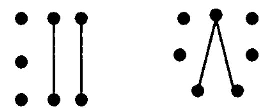


Figure 3.1

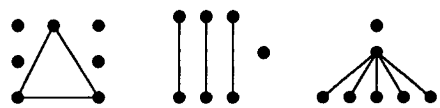
We now present our main results, with graphs G_i below being identified by their complements. Our reasoning and computations will be presented in subsequent sections.

Theorem 3.1. *With the graphs G_1 through G_{36} identified below by their complements, the Ramsey numbers $R(3, G)$ for all connected graphs G, $n(G) = 7$, are determined as follows:*

- $R(3, K_7) = 23$.
- $R(3, K_7 - e) = 21$.
- *Let* $T = \{G_1, G_2\}$, $B = \{G_3, G_4, G_5\}$.
  $R(3, G) = 18$ *if and only if* $\exists P \in T, \exists Q \in B$, *such that* $Q \subseteq G \subseteq P$.

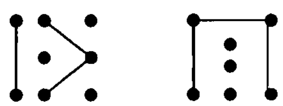


$\overline{G_1}$ $\overline{G_2}$

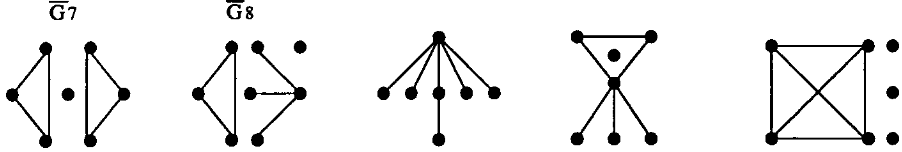


$\overline{G_3}$ $\overline{G_4}$ $\overline{G_5}$

- *Let* $T = \{G_6, G_7\}$, $B = \{G_8, G_9, G_{10}, G_{11}, G_{12}\}$.
  $R(3, G) = 17$ *if and only if* $\exists P \in T, \exists Q \in B$, *such that* $Q \subseteq G \subseteq P$.

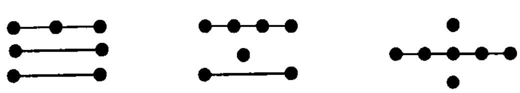


$\overline{G_7}$ $\overline{G_8}$

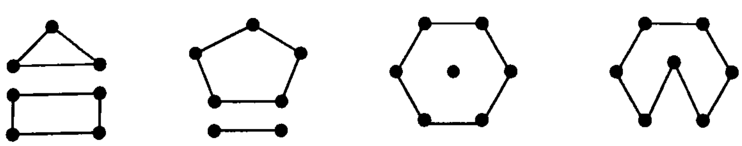


$\overline{G_9}$ $\overline{G_{10}}$ $\overline{G_{11}}$ $\overline{G_{12}}$ $\overline{G_{13}}$

- *Let* $T = \{G_{13}, G_{14}, G_{15}\}$, $B = \{G_{16}, G_{17}, G_{18}, G_{19}\}$.
  $R(3, G) = 16$ *if and only if* $\exists P \in T, \exists Q \in B$ *such that* $Q \subseteq G \subseteq P$.

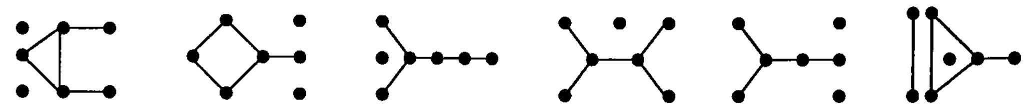


$\overline{G_{13}}$ $\overline{G_{14}}$ $\overline{G_{15}}$

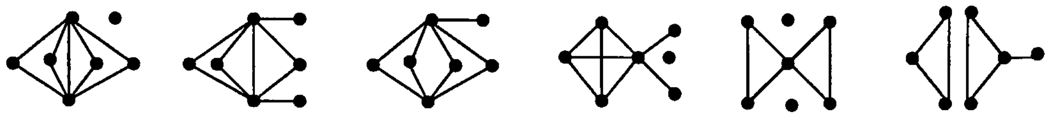


$\overline{G_{16}}$ $\overline{G_{17}}$ $\overline{G_{18}}$ $\overline{G_{19}}$

- Let $T = \{G_{20}, G_{21}, G_{22}, G_{23}, G_{24}, G_{25}\}$, $B = \{G_{26}, G_{27}, G_{28}, G_{29}, G_{30}, G_{31}\}$. $R(3, G) = 14$ *if and only if* $\exists P \in T, \exists Q \in B$, *such that* $Q \subseteq G \subseteq P$.

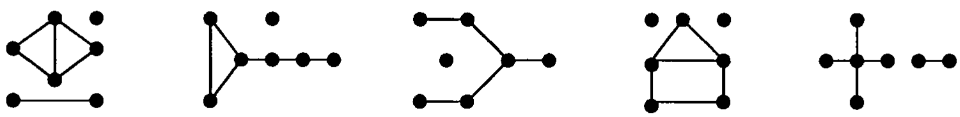


$\overline{G}_{20}$ $\overline{G}_{21}$ $\overline{G}_{22}$ $\overline{G}_{23}$ $\overline{G}_{24}$ $\overline{G}_{25}$



$\overline{G}_{26}$ $\overline{G}_{27}$ $\overline{G}_{28}$ $\overline{G}_{29}$ $\overline{G}_{30}$ $\overline{G}_{31}$

- Let $T = \{G_{32}, G_{33}, G_{34}, G_{35}, G_{36}\}$. $R(3, G) = 13$ *if and only if* $\exists P \in T$, *such that* $G \subseteq P$.



$\overline{G}_{32}$ $\overline{G}_{33}$ $\overline{G}_{34}$ $\overline{G}_{35}$ $\overline{G}_{36}$

- *No connected graph* G *on* 7 *vertices has Ramsey number* $R(3, G) = 22, 20, 19, 15$.

Table 3.1 lists the number of graphs versus their Ramsey numbers.

Ramsey number	number of graphs
23	1
21	1
18	7
17	19
16	16
14	40
13	769
all others	0

Table 3.1. Ramsey numbers r versus number of connected graphs G with $R(3, G) = r, n(G) = 7$

4. From theory to algorithms

Traditionally, it is agreed upon that a proposition is *proved* if it can be deduced from only assumed facts and conditions, in a finite number of steps, with each step applying axioms of logic and inference rules. However, there are problems whose proofs are so lengthy and tedious that it is very hard to complete them, such as the famous four-color problem. In these cases, machines are used to speed up the process of reasoning provided that the algorithms in use are well founded and properly verified. We consider this computer aided approach also a *proof.* The current trends are that computers will play more and more active role in fields which used to belong solely to pure mathematics.

In this section we develop schemes to prove the theorems from previous section. Since we can obtain all $(3,6)$-good graphs, we first consider generally what we can deduce from those graphs; then, for the remaining cases, develop specific lemmas and algorithms to obtain Ramsey numbers and critical graphs.

4.1. General cases

We split all connected graphs on 7 vertices into different categories with respect to their minimum degree, $\delta(G)$. In this subsection we consider the cases where $\delta(G) \leq 3$ and $\delta(G) = 4$. Graphs with $\delta(G) = 5$ will be analyzed in the next subsection.

Theorem 4.1. *For any connected graph G on 7 vertices, $R(G,3) \geq 13$.*

Proof:

It is obvious that a graph formed by two disjoint copies of K_6 is $(G,3)$-good, for any connected graph G on seven vertices; hence $R(G,3) \geq 13$. ■

Theorem 4.2. *For any connected graph G with $n(G) = 7$, any $(G,3)$-good graph containing no K_6 is included in the base of $(6,3)$-good graphs. Furthermore, if $R(G,3) > 13$, this graph data base is sufficient to determine $R(G,3)$ for the case $\delta(G) \leq 3$.*

Proof:

The first part is obvious by definition.

For the second part, assume the contrary. Let F be $(G,3)$-good, where $\delta(G) \leq 3$, $n(F) \geq 13$, and F contains a K_6 with vertex set X. There are two cases:

(i) $\exists v_1, v_2 \in V(F) - X$ such that $(v_1, v_2) \notin E(F)$. To avoid an independent set of size 3 among v_1, v_2 and X, it is necessary that $|N_X(v_1)| + |N_X(v_2)| \geq 6$. So $\delta(G) \leq 3$ implies that $N_X(v_1) \cap N_X(v_2) = \emptyset$, and hence $|N_X(v_1)| = |N_X(v_2)| = 3$. But F still contains G, which is a contradiction.

(ii) $V(F) - X$ induces a clique in F. $n(G) = 7$ requires $|V(F) - X| \leq 6$, hence $n(F) \leq 12$; contradiction. ∎

The method to generate all $(6,3)$-good graphs has been described before by Radziszowski and Kreher in [RK1]. Here we generated independently all 761,692 $(6,3)$-good graphs. We used a quite efficient algorithm by McKay [Mc] to delete isomorphic copies of a given graph. The generation of all non-isomorphic $(6,3)$-good graphs is important in this case, not only because it is an excellent base to filter $(G,3)$-good graphs for $n(G) = 7$, but it also enables us to catalog all $(3, K_6 - 2K_2)$-good graphs, which will help us in the special cases of $R(3, K_7 - 2K_2)$ and $R(3, K_7 - 3K_2)$. We also generated all 1044 non-isomorphic graphs on 7 vertices; 853 of them are connected.

We now come to the most important remaining task in this case — generate as many as possible of the critical graphs for all 853 connected graphs on 7 vertices. (Due to the limitations of Theorem 4.2, we cannot guarantee that all critical graphs can be found for some graphs G with $\delta(G) \geq 4$.) This goal is unreachable without the assistance of computers. Fortunately, we know that if $G_1 \subseteq G_2$, then all $(G_1, 3)$-good graphs must also be $(G_2, 3)$-good. Therefore we decompose the task into two subtasks.

(i) Generate all containment relations for all connected graphs on 7 vertices. It is obvious that the containment relation is actually a partial order. It is produced by the procedure described in Algorithm *makeNet*.

(ii) Examine all $(6,3)$-good graphs, scanning through the lattice we built in step (i) to find $(G,3)$-good graphs of the maximum order for each connected graph on 7 vertices, as described in Algorithm *scanNet*.

If we denote a containment relation by a direct edge between nodes, after calling $makeNet(\Gamma, 7)$, Γ will actually be the *Hasse diagram* for partial ordering with the relation of containment over all graphs on 7 vertices, since any two graphs can only differ one edge

procedure *makeNet*(N, K)

— N is a net, K is an integer

— each node in N corresponds to a connected graph on 7 vertices

for each connected graph G on K vertices

locate node n in net N corresponding to graph G

let $(e_1, e_2, \cdots, e_k)$ be a list of all edges of G

for $i := 1$ **to** k **do**

delete edge e_i from G to obtain graph G_i

if G_i is connected **then**

locate n_i in net N corresponding to graph G_i

create a directed edge in net N from node n to node n_i

endif

endfor

endfor

end *makeNet*

Algorithm 4.1. *makeNet*

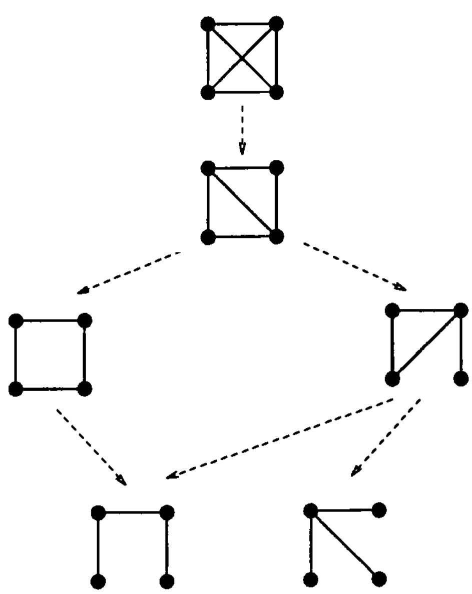


Figure 4.1

if they are related in the Hasse diagram.

As an example, Figure 4.1 presents a net of all connected graphs on 4 vertices, with the dotted arrows representing the containment relation.

There are some details of *scanNet* that need to be explained. It is very important that in line (*) of Algorithm *scanNet* we match a graph with the corresponding node in net. For this purpose we used canonical labeling, produced by a sophisticated graph isomorphism tool, *"nauty"* [Mc].

If N is the net after calling procedure $makeNet(N, 7)$, $scanNet(N, 7)$ will produce critical $(G, 3)$-good graphs which are present in the base of all $(6, 3)$-good graphs.

Now by Theorem 4.2 we need to consider only the case of all connected graphs G of order 7 with minimum degree 4, such that there exists a $(G, 3)$-good graph F containing a K_6.

Theorem 4.3. *For any connected graph G of order 7 with $\delta(G) = 4$, if F is a critical $(G, 3)$-good graph containing two disjoint K_6, then F must be unique: 5 triangles forming a pentagon, with 9 edges between each pair of neighboring triangles, as shown in Figure 4.2.*

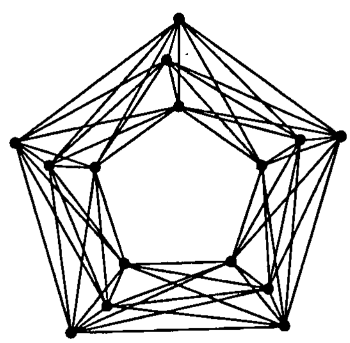


Figure 4.2

Proof:

Let X_1, X_2 be two disjoint vertex sets of K_6 in F, and $Z = V(F) - X_1 - X_2$. Let z be any vertex in Z. Also define vertex sets $Y_i = X_i - N_{X_i}(z)$, $\quad i = 1, 2$. Since $\delta(G) = 4$, $(G, 3)$-good graph F must satisfy $d(z, X_1) \leq 3$ and $d(z, X_2) \leq 3$, i.e., $|Y_1|, |Y_2| \geq 3$. To

```
procedure scanNet(N, K)
    — N is a net, K is an integer.
    for m := R(6,3) − 1 downto 13 do
        let (F_1, F_2, ···, F_k) be a list of all (6,3)-good graphs on m vertices
        for i := 1 to k do
            initialize all nodes of net N to be unmarked
            let (S_1, S_2, ···, S_p) be a list of all K-subsets of V(F_i)
            for j := 1 to p do
                G := graph induced by S_j in F_i
                if G is not connected then continue endif
                let n be the node in net N corresponding to G.          (*)
                if G is unmarked then
                    mark node n and all its children
                endif
            endfor
            {assert : ∀ unmarked node n in net N, with n corresponding
                to G, F_i is (G,3)-good.}
            for each node n in net N, n corresponding to graph G' do
                if n is unmarked then
                    {assert : F_i is (G',3)-good.}
                    if there is no (G',3,m+1)-good graph stored then
                        store F as a (G',3,m)-good graph.
                    endif
                endif
            endfor
        endfor
    endfor
end scanNet
```

Algorithm 4.2. *scanNet*

avoid an independent set of size 3 between Y_1, Y_2 and z, $Y_1 \cup Y_2$ will induce a complete graph. Hence $|Y_1| = |Y_2| = 3$.

Now consider any vertex $z' \in Z$, $z' \neq z$. To avoid an independent set of size 3, the condition that there is no edge between $N_{X_1}(z)$ and Y_2 enforces that z' must be fully connected to $N_{X_1}(z)$. For the same reason, z' must be fully connected to $N_{X_2}(z)$; these two facts imply that Z is fully connected to $N_{X_i}(z)$, and there is no edge between Z and $Y_1 \cup Y_2$. Hence $Z \cup N_{X_i}(z)$ induces a complete graph, $i = 1, 2$.

In order for F to be critical, $|Z| = 3$. Therefore the structure of F is determined: A graph of order 15 containing 5 vertex disjoint triangles, X_1, X_2, Y_1, Y_2, Z, arranged in a pentagon, with 9 edges between each pair of neighboring triangles, as shown in Figure 4.2.

■

Hence for connected graph G with $n(G) = 7$ and $\delta(G) = 4$, it is sufficient to consider $(G, 3)$-good graph F such that $n(F) \geq 15$, F contains a vertex subset X inducing a K_6, but F does not contain two vertex disjoint K_6's.

Define vertex sets $S = \{s \in V(F) - X : d(s, V(F) - X - s) = |V(F) - X| - 1\}$, and $Y = V(F) - X - S$. Obviously, $F[S]$ is complete, $F[Y]$ is not complete.

Algorithm 4.3 on next page constructs a family of vertex sets Y_{ij}, for $i = 1, 2, \cdots, k$. The purpose of this algorithm is to provide a concise description for some theoretical constructions; it is not intended for an implementation on computers. The construction described by the algorithm is non-deterministic, and any instance of the construction will suffice for our reasoning later.

Let $Y_j = \bigcup_i Y_{ij}$, $j = 1, 2$. Obviously, Y_1, Y_2 is a partition of Y, and $\forall y \in Y$, $d(y, X) = 3$. Also, Y_j induces a complete graph in F, $j = 1, 2$.

Theorem 4.4. *For a connected graph G of order 7 with minimum degree 4, there is no $(G, 3)$-good graph F such that $n(F) \geq 16$, F contains a K_6, but F does not contain two vertex disjoint K_6.*

Proof:

Assume the contrary: let F be a $(G, 3)$-good graph, and $n(F) \geq 16$. Let $X \subset V(F)$ such that X induces a K_6 in F. Define vertex sets $S = \{s \in V(F) - X : d(s, V(F) - X - s) = |V(F) - X| - 1\}$, and $Y = V(F) - X - S$. We have $|S \cup Y| \geq 10$, and subgraph induced

```
Z := Y;
i := 1;
while Z ≠ ∅ do
begin
    { assert: F[Z] is not a clique }
    let (z1, z2) be a non-edge in E(F);
    Yi1 := {z1};        Yi2 := {z2};
    Z := Z − {z1, z2};
    for each z ∈ Z do
    { assert: (z, z1) ∈ E(F) ∨ (z, z2) ∈ E(F)}
    begin
        if (z, z1) ∉ E(F) then
            Yi2 := Yi2 ∪ {z}
            Z := Z − {z}
        elseif (z, z1) ∉ E(F) then
            Yi1 := Yi1 ∪ {z}
            Z := Z − {z}
        endif
    endfor
    i := i + 1
endwhile
```

Algorithm 4.3

by $S \cup Y$ does not contain K_6. Noting that $|S| \leq 5$ since $F[S]$ is complete, we consider the order of S separately as follows:

i) $4 \leq |S| \leq 5$.

In this case $|Y| \geq 5$. Since Y cannot be an independent set, any edge together with S will induce a K_6 in F disjoint from X. Contradiction.

ii) $|S| = 3$, $|Y| \geq 7$.

Since $R(3,3) = 6$, Y must contain a triangle if an independent set of 3 is to be avoided, and this triangle combined with S will yield to a K_6 in F disjoint from X. Contradiction.

iii) $|S| = 2$, $|Y| \geq 8$.

Let us partition Y into sets Y_1, Y_2 according to Algorithm 4.3. Either Y_1 or Y_2 will induce a K_4, together with S, inducing another K_6 in F. Contradiction.

iv) $|S| = 1$, $|Y| \geq 9$.

Partition Y into sets Y_1, Y_2 according to Algorithm 4.3. Either Y_1 or Y_2 induces a K_5, together with S, resulting a K_6. Contradiction.

v) $S = \emptyset$, $|Y| \geq 10$.

Partition Y into sets Y_1, Y_2 according to Algorithm 4.3. The only possible case is that both Y_1 and Y_2 are K_5's, and there is no K_6 in $F[Y]$. Since $\forall y \in Y$, $d(y, X) = 3$, there are 30 edges between X and Y in graph F. Hence $\forall x \in X$, $d(x, Y) = 5$. Without loss of generality, assume that $\exists x \in X$, $d(x, Y_1) = 5$. Furthermore, since there are at least 3 pairs of components, (otherwise, a triangle component in Y_2 with corresponding triangle connection to X will be another K_6, resulting two disjoint K_6) at least one pair of components Y_{i1}, Y_{i2} are single vertices, therefore $F[\{x\} \cup Y_1 \cup Y_{i2}]$ will contain $K_7 - K_{1,2}$, and F cannot be $(G, 3)$-good. Contradiction. ■

Note that Theorem 4.3 and Theorem 4.4 include all the cases for $\delta(G) = 4$. If we know that after the *scanNet* operation, there is no $(G, 3, n)$-good graph in the base of $(6, 3)$-good graphs with $n \geq 15$, then the graph shown in Figure 4.2 is $(G, 3)$-critical and $R(3, G) = R(G, 3) = 16$.

4.2. Special cases

Now the only cases left to be studied are the connected graphs on 7 vertices which have minimum degree of 5 or 6: namely K_7, $K_7 - e$, $K_7 - 2K_2$ and $K_7 - 3K_2$. K_7 is the classical case — $R(3, 7) = 23$ [GY], and it has 191 critical graphs [RK1]. It was determined in [GH] that

$$R(3, K_7 - e) = 21,$$

and the critical graph is unique [Ra1]. Below we devise specialized algorithms for Ramsey numbers $R(3, K_7 - 2K_2)$ and $R(3, K_7 - 3K_2)$, and corresponding critical graphs.

Theorem 4.5. (variation of proposition 4 in [GY]) *For any $(3, K_r - iK_2, n)$-good graph G with e edges,*

$$\Delta = ne - \sum_{j=0}^{r-1} n_j \left(e(3, K_{r-1} - iK_2, n - j - 1) + j^2 \right) \geq 0, \tag{2}$$

where $i \leq \lfloor \frac{r-1}{2} \rfloor$, n_j is the number of vertices of degree j in G, $n = \sum_{j=0}^{r-1} n_j$ and $2e = \sum_{j=0}^{r-1} j \cdot n_j$.

Proof:

Given $(K_3, K_r - iK_2, n)$-good graph G, for any vertex v in G, let $H(v)$ be the graph induced by $V(G) - \{v\} - N(v)$ in G. Since $N(v)$ induces an independent set in G, the number of edges satisfies

$$e = e(H(v)) + \sum_{w \in N(v)} deg(w). \tag{3}$$

If we sum (3) over all vertices of G, we have

$$ne = \sum_{v \in V(G)} e(H(v)) + \sum_{v \in V(G)} \sum_{w \in N(v)} deg(w). \tag{4}$$

Since $H(v)$ is $(3, K_{r-1} - iK_2, n - deg(v) - 1)$-good, we have

$$e(H(v)) \geq e\left(3, K_{r-1} - iK_2, n - deg(v) - 1\right). \tag{5}$$

On the other hand,

$$\begin{aligned} \sum_{v \in V(G)} \sum_{w \in N(v)} deg(w) &= \sum_{v,w:(v,w) \in E(G)} deg(w) \\ &= \sum_{w \in V(G)} (deg(w))^2 \\ &= \sum_{j \geq 0} n_j j^2. \end{aligned} \tag{6}$$

Hence using (5) and (6) in (4) gives

$$ne \geq \sum_{j=0}^{r-1} n_j \left(e(3, K_{r-1} - iK_2, n - j - 1) + j^2 \right).$$

which clearly is equivalent to (2). ∎

Theorem 4.5 provides a theoretical basis for linear programming, which often proves useful for estimating the upper bound of Ramsey numbers. It usually requires the calculation of $e(3, G, n)$ — the minimal number of edges for triangle-free graphs of prescribed size. We tabulate all $(3, K_6 - 2K_2, n, e)$-graphs in Table 4.1. From the table we know that $e(3, K_6 - 2K_2, 12) = 24$, and $e(3, K_6 - 2K_2, 11) = 18$. Theorem 4.6 is an immediate application of this technique.

Theorem 4.6. $R(K_7 - 2K_2, 3) \leq 19$.

Proof:

Assume the contrary, and let F be a $(3, K_7 - 2K_2, 19)$-good graph. Apparently $n = n_6 = 19$ in this case. Applying Theorem 4.5 we have

$$n_6 = 19,$$

$$2 \cdot e = 6 \cdot 19,$$

$$\Delta = 19 \cdot e - n_6 \cdot (24 + 36) \geq 0,$$

which is a contradiction. ∎

We now develop some theorems to facilitate the design of algorithms for the cases of $K_7 - 2K_2$ and $K_7 - 3K_2$.

Theorem 4.7. *Let F be a $(3, K_r - iK_2)$-good graph, and x be a vertex of the minimum degree in F. Define $H(x) = V(G) - x - N(x)$. Let g be any vertex in $N(x)$, and consider the set of all vertices of $H(x)$ which are connected to g. The following two conditions must be met by $N_{H(x)}(g)$.*

1) *The graph induced by $N_{H(x)}(g)$ in F is an independent set.*
2) *The graph induced by $H(x) - N_{H(x)}(g)$ in F is $(3, K_{r-2} - (i-1)K_2)$-good.*

Proof:

1) is true because otherwise a triangle is created.

2) is true because otherwise $\left(H(x) - N_{H(x)}(g)\right) \cup \{x, g\}$ will induce a graph whose complement contains $K_r - iK_2$. ∎

edges (e)	3	4	5	6	7	8	9	10	11	12	all n
0	1	1	1								3
1	1	1	1								3
2	1	2	2	1							6
3		2	3	4							9
4		1	4	7	2						14
5			2	9	9						20
6			1	7	18	1					27
7				4	20	10					34
8				2	18	32	1				53
9				1	11	57	4				73
10					5	57	18				80
11					1	38	70				109
12					1	21	129	1			152
13						9	129	9			147
14						3	87	5			144
15						2	42	146			190
16						1	20	199			220
17							8	168			176
18							3	99	5		107
19							1	49	23		73
20							1	22	51		74
21								10	50		60
22								4	30		34
23								2	11		13
24								1	4	1	6
25								1	1	3	5
26										5	5
27										4	4
28										2	2
29										1	1
total	3	7	14	35	85	231	513	765	175	16	1844

Table 4.1. The number of $(3, K_6 - 2K_2, n, e)$ graphs by number of edges e and vertices n

Based on those conditions, given $H(x)$, it is feasible to generate all possible sets $N_{H(x)}(g)$. We call them *cones* to delineate the fact that $N_{H(x)}(g)$ is formed by all vertices in $H(x)$ connected to g.

Given that x is a vertex with minimum degree δ of a $(3, K_r - iK_2)$-good graph F, $H(x)$ is $(3, K_{r-1} - iK_2)$-good. we use the procedure described in Algorithm *genCone* to generate all possible cones for vertices in $N(x)$. Upon completion of *genCone*, *List* contains the set of all possible cones for vertices in $N(x)$.

> **procedure** $genCone(H, \delta, List)$
>
> — H is $(3, K_{r-1} - iK_2)$-good
>
> $List := \emptyset$
>
> **foreach** $S \subseteq H$
>
> **if** S satisfies
>
> (i) $\delta - 1 \leq |S| \leq r - 2$
>
> (ii) S induces an independent set in H
>
> (iii) $V(H) - S$ does not contain $K_{r-2} - (i-1)K_2$ in complement.
>
> **then** $List := List \cup \{S\}$
>
> end *genCone*

Algorithm 4.4. *genCone*

We know that x is fully connected with an independent set $N(x)$. However, we need to consider further the relations among cones to determine the structure of F. Here we use a technique that proved successful for many small Ramsey number cases, as illustrated by McKay and Zhang [MZ].

Suppose F is a $(3, K_r - iK_2)$-good graph, x is a vertex of minimum degree in F, $d(x) = \delta$, and let $S_1, S_2, ..., S_N$ be a list of cones in $H(x)$, produced by *genCone*, with cardinality between $\delta - 1$ and $r - 2$, inclusive. Let w be a vertex in $H(x)$, and $X_1, X_2, ..., X_i \subseteq H(x)$ (cones). Define

$$d_i(w) = d_i(w; X_1, ..., X_i) = deg_H(w) + |\{X_j | w \in X_j, 1 \leq j \leq i\}|,$$

and consider the recursive procedure described in Algorithm *makeX*.

procedure $makeX(k, (X_1, ..., X_{k-1}), (Y_1, ..., Y_K))$

– k and K are integers. Each X_i and each Y_i is a subset of $H(x)$.

if $k > \delta$ **then**

$process((X_1, X_2, ..., X_\delta))$

else

Construct the list $(Z_1, ..., Z_L)$ of all elements Z of $\{Y_1, ..., Y_K\}$ such that

(i) For each $w \in H(x)$, if $d_{k-1}(w) < k - 1$, then $w \in Z$.

(ii) For each $w \in H(x)$, if $d_{k-1}(w) = r - 1$, then $w \notin Z$.

(iii) H has no independent set of size $r - 1 - |I|$ disjoint from $Z \cup \bigcup_{i \in I} X_i$ for any $I \subseteq \{1, 2, ..., k-1\}$.

for $i := 1$ **to** L **do**

$makeX(k + 1, (X_1, ..., X_{k-1}, Z_i), (Z_i, Z_{i+1}, ..., Z_L))$

endif

end $makeX$.

Algorithm 4.5. *makeX*

We applied a technique similar to [MZ], in which it is proved that

Theorem 4.8. [MZ] *Suppose procedure makeX is invoked with* $(1, (), (S_1, ..., S_N))$ *as arguments. Then procedure process will be invoked exactly once for each sequence* $X_1 = S_{i_1}, X_2 = S_{i_2}, \cdots, X_\delta = S_{i_\delta}$ *such that* $1 \le i_1 \le i_2 \le \cdots \le i_\delta \le N$ *and the conditions in Theorem 4.7 are met, and for no other sequences.*

This procedure provides a feasible way to determine all possible combinations of cones for all vertices in $N(x)$, hence makes the determination of $(3, K_r - iK_2)$-good graph F possible.

Theorem 4.9. $R(K_7 - 2K_2, 3) \le 18$.

Proof:

Assume contrary, let F be a $(3, K_7 - 2K_2, 18)$-good graph. $R(3, K_6 - 2K_2) = 13$

requires that $n_4 = 0$. Applying Theorem 4.5, we have

$$n_5 + n_6 = 18,$$

$$2 \cdot e = n_5 \cdot 5 + n_6 \cdot 6,$$

$$\Delta = 18 \cdot e - n_5 \cdot (24 + 25) - n_6 \cdot (18 + 36) \geq 0,$$

which implies $n_5 = 0$ and $n_6 = 18$. Hence any $(3, K_7 - 2K_2, 18)$-good graph must be regular of degree 6, i.e., $H(x)$ must be $(3, K_6 - 2K_2, 11, 18)$-good. There are exactly 5 such graphs. (See Table 4.1.) There is only 1 cone for one of the five graphs, which makes further gluing of $N(x)$ impossible. Hence the theorem holds. ∎

Theorem 4.10. *$R(3, K_7 - 2K_2) = 18$, and there is a unique $(3, K_7 - 2K_2, 17)$-good graph, as shown in Figure 4.3.*

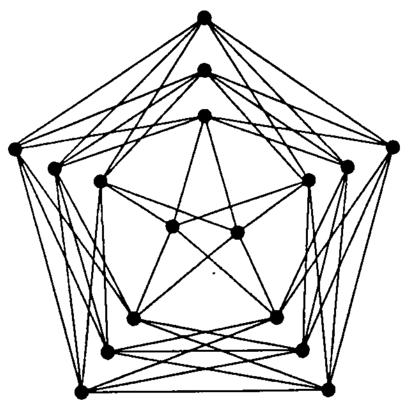


Figure 4.3

Proof:

We first prove that $R(3, K_7 - 2K_2) \geq 18$ by constructing all $(3, K_7 - 2K_2, 17)$-good graph F, which turns out to be unique. Let x be a vertex of minimum degree in F. Since $H(x)$ is $(3, K_6 - 2K_2)$-good, $R(3, K_6 - 2K_2) = 13$ requires that $n_3 = 0$. We consider the degree of x in F:

(i) $d(x) = |N(x)| = 4$.

We generate cones with size at least 3 by calling $genCode(H(x), 3, List)$, where $H(x)$ is every $(3, K_6 - 2K_2)$-good graph on 12 vertices. Using parameters $\delta = 4$, $r = 7$, we ran procedure $makeX(1, (), (S_1, S_2, ..., S_N))$, $(S_1, S_2, ..., S_N)$ being the list of cones generated before. Results indicated that there was no such possible $(3, K_7 - 2K_2)$-good graph F.

(ii) $d(x) = |N(x)| = 5$.

Calling $genCode(H(x), 3, List)$, where $H(x)$ is every $(3, K_6 - 2K_2)$-good graph on 11 vertices, and $makeX(1, (), (S_1, S_2, ..., S_N))$, where $\delta = 5, r = 7$ and $(S_1, S_2, ..., S_N)$ is the list of cones generated before, generated one $(3, K_6 - 2K_2)$-good graph, as shown in Figure 4.3.

(iii) $d(x) = |N(x)| = 6$.

Calling $genCode(H(x), 5, List)$, where $H(x)$ is every $(3, K_6 - 2K_2)$-good graph on 10 vertices, and $makeX(1, (), (S_1, S_2, ..., S_N))$, where $\delta = 6, r = 7$ and $(S_1, S_2, ..., S_N)$ is the list of cones generated before, indicates there is no such $(3, K_7 - 2K_2)$-good graph.

Clearly (i) (ii) and (iii) exhaust all possibilities, hence by Theorem 4.9, this theorem is established. ∎

Similarly, we can use this approach to calculate $R(3, K_7 - 3K_2)$.

Theorem 4.11. *$R(3, K_7 - 3K_2) = 18$, with a unique critical graph as shown in Figure 4.3.*

Proof:

Since any $(3, K_7 - 3K_2)$-good graph does not contain $K_7 - 2K_2$ in its complement, hence it is also $(3, K_7 - 2K_2)$-good. Therefore by Theorem 4.10, $R(3, K_7 - 3K_2) \leq 18$.

We checked by hand, and verified by computer, that the unique $(3, K_7 - 2K_2)$-good graph on 17 vertices shown in Figure 4.3 is also $(3, K_7 - 3K_2)$-good, hence the theorem holds. ∎

This completes our derivation of Ramsey numbers $R(3, G)$, for all connected graph G on 7 vertices.

5. Computations and verification

The successful evaluation of Ramsey numbers for our graphs depended on a careful implementation and verification of computer programs. We first obtained all non-isomorphic graphs up to order 8 using simple vertex-extension algorithms. By counting those graphs and checking with previous catalogues, we confirmed that all graphs up to order 8 were obtained. By recurrence relation, we determined that there are altogether 853 non-isomorphic connected graphs on 7 vertices. Our results obtained using computer verified this calculation.

All $(3,6)$-good graphs were generated before by McKay and Radziszowski independently. In this project, we generated all $(6,3)$-good graphs again, independently. The total number of these graphs as well as the cataloging of graphs by number of edges and vertices agreed with previous publications [RK1]. This was a strong evidence that we obtained the complete $(6,3)$-good graph base.

Due to the large number of $(6,3)$-good graphs (7×10^5), it was necessary to distribute the calculation of Ramsey numbers for all 853 graphs. We utilized about 20 Sun workstations (mostly Sparc SLC) simultaneously. The total amount of computations involved was approximately 1200 hours of CPU time.

After obtaining the $(6,3)$-good graph base, we ran the same program of *scanNet* on all connected graphs of order 6. Our computation agreed with the results obtained by Faudree, Rousseau and Schelp [FRS]. Furthermore, we obtained all corresponding critical Ramsey graphs. It is interesting to see that besides a unique critical graph for all graphs having Ramsey number $R(3,G) = 14$ (See [FRS]), there is also a unique critical graph for all graphs having Ramsey number $R(3,G) = 12$, shown in Figure 5.1.

The set of programs to construct $(3, K_7 - 2K_2)$- and $(3, K_7 - 3K_2)$-good critical graph was checked by repeating the same computation to construct $(3, K_6 - e)$- and $(3, K_6 - 3K_2)$-good graphs from all $(5,3)$-good graphs. The number of critical graphs obtained here agreed with previous construction from *scanNet*. Hence we are highly confident that the results are correct.

It is worth mentioning that most connected graph G on 7 vertices having Ramsey

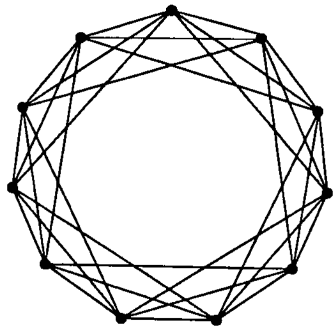


Figure 5.1

number $R(3,G) = 14$ have a unique $(G,3)$-good graph on 13 vertices with regular degree 8, shown in Figure 5.2.

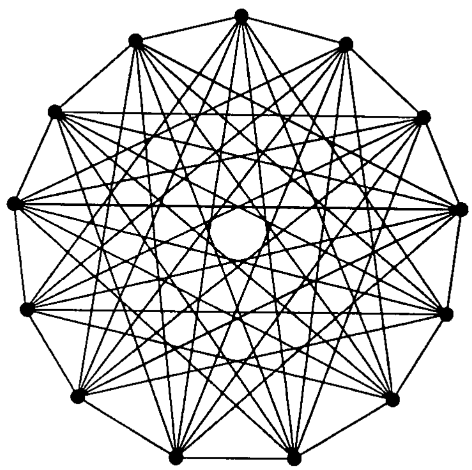
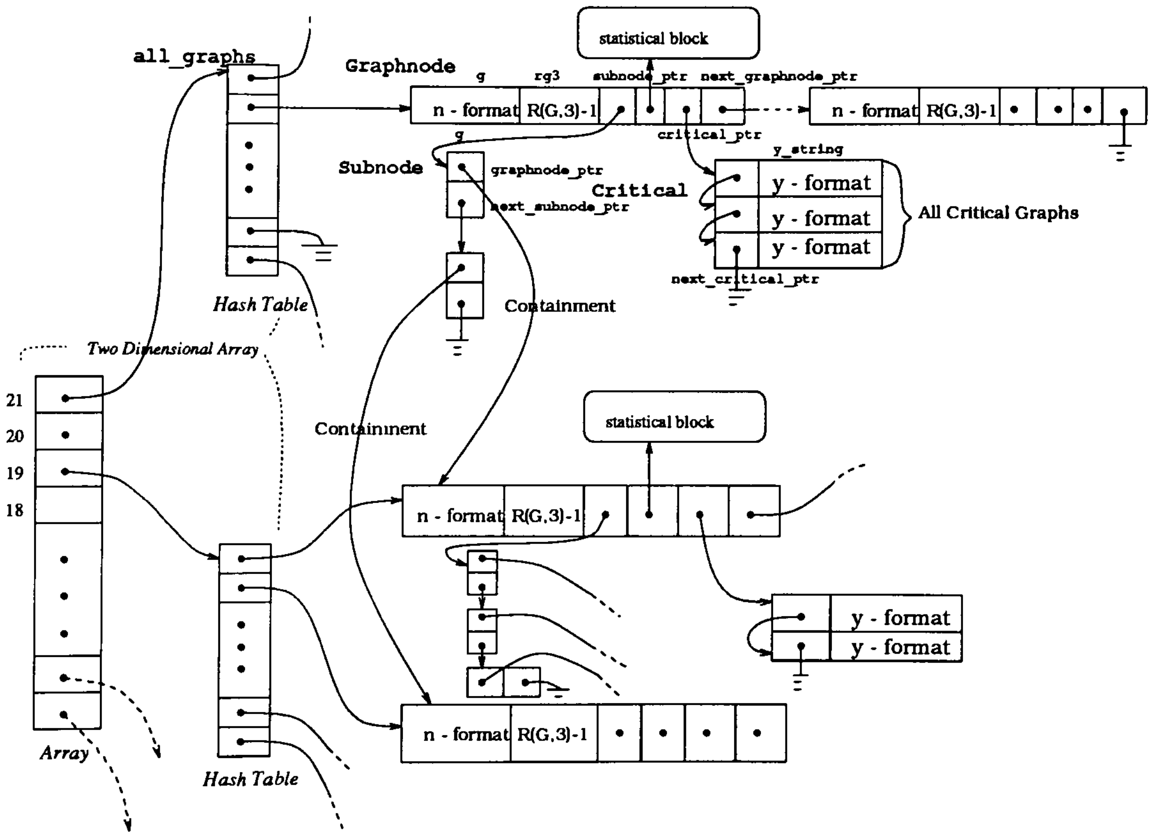


Figure 5.2

We feel that it is also worthwhile to mention below some primary data structures used in this work, which may have wider applicability.

We used two forms to represent a graph. Adjacency matrix representation (called *n-format*) was suitable for a small-sized graph, and was used here to carry out various graph manipulations such as canonical labeling [Mc], complement, edge vertex count, subgraph enumeration and testing. Another form — called *y*-format — was used for storage of most graphs. Because of its compactness and usefulness, we explain it here in some detail: Each graph occupies one line with a terminating newline. Except for the newline, each byte has the format `01xxxxxx`, where each `x` represents one bit of data. The first byte `xxxxxx` is the number of vertices n. Other $\lceil \frac{n(n-1)}{12} \rceil$ bytes contain the upper triangle of the adjacency matrix in column major order. The bits are used in left to right order within each byte. Any unused bits on the end are set to zero. y-format can handle graphs of no more than 63 vertices, which is enough for the purpose of this work. It provided a simple encoding scheme with mostly printable characters.

To facilitate the *scanNet* operation described in section 4.1, we implemented a general purpose net structure, to characterize all connected graphs on 7 vertices. The general picture of the net is depicted in Figure 5.3.



Data Structures for R(G,3) Calculation

Figure 5.3

Nodes in the net were split into different layers, each layer having the property that nodes of that layer can only be siblings of one another. All nodes could have multiple parents and multiple children among neighboring layers, which obey a partial order relation. Every node in the net could be accessed and located via a fast two dimensional hash table, with each layer represented by a one dimensional hash table. Each node had various fields to describe its unique information. Since it was anticipated that there are not too many $(G, 3, R(G, 3) - 1)$-good graphs for each graph G on 7 vertices, we just made a simple linked list of these y-format graphs for each G.

It is also important to note that in this study, we used a quite powerful tool for graph isomorphism — program *nauty*, written and distributed by Brandan McKay [Mc]. It is essential that we can efficiently delete isomorphic copies of a graph in the middle of computation. Without the tremendous help of this software, the whole computation would be much harder.

6. What next?

There are other related results obtained through this work. First of all, we obtained not only triangle Ramsey numbers for connected graphs of order 7, but also most of the corresponding critical graphs. Our selective search is very effective to obtain all critical graphs for the majority of cases; the only missing critical graphs are possibly those on 15 vertices containing one K_6, but do not contain two disjoint copies of K_6. Those critical graphs contain rich information with regard to the fundamental property of a graph.

Secondly, recall the definition of functions $f(k,n)$ and $g(k,n)$ defined in Section 2.3. It is easy to see from Theorem 3.1 that $R(3,G_{26}) = 14$, and $e(G_{26}) = 12$, and all connected graphs G on 7 points and 11 edges have Ramsey numbers $R(3,G) = 13$, hence $f(3,7) = 11$. It is obvious from Theorem 3.1 that G_{34} and G_{36} on 7 points and 16 edges are 3-good. Therefore $g(3,7) = 16$.

There are quite interesting extensions related to our results. On one hand, there can be some very productive work along the theme of 3-goodness by applying our new results on 7 vertices. Theorem 2.4 and Theorem 2.5 served as a starting point for the study of general k-goodness. Theorem 2.7 gives a lower bound of $f(3,n)$, which is still acceptable for $n = 6$, but not satisfactory for $n = 7$. It is quite probable that by observing some relations among graphs described in Section 3, a better general lower bound can be found.

On the other hand, our computational methods might be widely applicable for the evaluation of various other Ramsey numbers. In particular, $R(3,K_{10}-e)$, $R(4,K_6-e)$ and $R(K_5-e,K_5-e)$ are suggested as some research problems.

Finally, we noticed that similar compilation for three-color Ramsey numbers for small graphs recently appeared [YR1] [YR2], (definition of multi-color Ramsey numbers can be found in [GRS]) which suggests that it is also a growing area.

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